where one now writes $\bar{w}$ for the new deflection vector. Eq. (8) can be written as

$$S\bar{w} = p - \Delta S\bar{w} \tag{9}$$

or

$$\bar{w} = F(p - \Delta S\bar{w}) \tag{10}$$

where $F = S^{-1}$.

To solve this equation for $\bar{w}$, an iteration procedure can be used:

$$w^{(v+1)} = F(p - \Delta S \cdot w^{(v)}) \tag{11}$$

with $w^{(0)} = w$ in Eq. (7). When ΔS is small, as when calculating an approximation of a partial derivative, the convergence is rapid. Note that the structural change term $-\Delta S w^{(v)}$ is treated as a load, which removes the necessity of recomputing the inverse of the altered structural stiffness matrix.

Thermoelastic Differential Equations for Shells of Arbitrary Shape

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DONNELL'S equations for the analysis of cylindrical shells have been extended to include the effects of arbitrary temperature distributions in Ref. 1. A special case of these equations are those for flat plates which are given in Ref. 2. Although there is no difficulty in further extending the treatment to shells of arbitrary shape, it is desirable to record for future reference the necessary formulation of the Vlasov-type equations for these problems, in view of the considerable interest in thermoelastic problems of shells.

Formulation of Thermoelastic Equations for General Shells

With the notation of Ref. 3, the five equations of equilibrium of a thin shell of arbitrary shape are given by

$$\frac{\partial(A_2T_1)}{\partial\alpha_1} + \frac{\partial(A_1S)}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2} S - \frac{\partial A_2}{\partial\alpha_1} T_2 = 0 \tag{1a}$$

$$\frac{\partial(A_2S)}{\partial\alpha_1} + \frac{\partial(A_1T_2)}{\partial\alpha_2} + \frac{\partial A_2}{\partial\alpha_1} S - \frac{\partial A_1}{\partial\alpha_2} T_1 = 0 \tag{1b}$$

$$\frac{1}{A_1A_2}\left[\frac{\partial(A_2N_1)}{\partial\alpha_1} + \frac{\partial(A_1N_2)}{\partial\alpha_2}\right] - \frac{T_1}{R_1} - \frac{T_2}{R_2} + q = 0 \tag{1c}$$

$$N_1 = \frac{1}{A_1A_2}\left[\frac{\partial(A_2M_1)}{\partial\alpha_1} + \frac{\partial(A_1H)}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2} H - \frac{\partial A_2}{\partial\alpha_1} M_2\right] \tag{1d}$$

$$N_2 = \frac{1}{A_1A_2}\left[\frac{\partial(A_2H)}{\partial\alpha_1} + \frac{\partial(A_1M_2)}{\partial\alpha_2} + \frac{\partial A_2}{\partial\alpha_1} H - \frac{\partial A_1}{\partial\alpha_2} M_1\right] \tag{1e}$$

where transverse shearing forces N_1 and N_2 and tangential surface forces; q_1 and q_2 have been disregarded in Eqs. (1a) and (1b).

The forces and moments are related to the strains and curvature changes of the middle surface by the following equations:

$$\begin{aligned}
T_1 &= \frac{E\delta}{1-\mu^2}(\epsilon_1 + \mu\epsilon_2) - \frac{N_T}{1-\mu} \\
T_2 &= \frac{E\delta}{1-\mu^2}(\epsilon_2 + \mu\epsilon_1) - \frac{N_T}{1-\mu} \\
S &= \frac{E\delta}{2(1+\mu)}\omega \\
M_1 &= \frac{E\delta^3}{12(1-\mu^2)}(k_1 + \mu k_2) - \frac{M_T}{1-\mu} \\
M_2 &= \frac{E\delta^3}{12(1-\mu^2)}(k_1 + \mu k_2) - \frac{M_T}{1-\mu} \\
H &= \frac{E\delta^3}{12(1+\mu)}\tau
\end{aligned} \tag{2}$$

where

$$N_T = \alpha E \int_{-\delta/2}^{\delta/2} T dz \qquad M_T = \alpha E \int_{-\delta/2}^{\delta/2} zT dz \tag{3}$$

where α is the coefficient of linear thermal expansion and T is the arbitrary temperature distribution at the point of interest.

When tangential displacement components in the equations for the changes of curvature and twist are neglected, the curvature change-displacement relations are given by

$$\begin{aligned}
k_1 &= -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right) - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} \\
k_2 &= -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right) - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} \\
\tau &= -\frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_2}\right)
\end{aligned} \tag{4}$$

The substitution of Eqs. (2) and (3) into Eqs. (1d) and (1e) and the neglect of some small terms (see Ref. 3, p. 86) yields

$$\begin{aligned}
N_1 &\cong -\frac{E\delta^3}{12(1-\mu^2)}\frac{1}{A_1}\frac{\partial\Delta w}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{M_T}{1-\mu}\right) \\
N_2 &\cong -\frac{E\delta^3}{12(1-\mu^2)}\frac{1}{A_2}\frac{\partial\Delta w}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{M_T}{1-\mu}\right) \\
\Delta &= \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\right)\right]
\end{aligned} \tag{5}$$

By introducing a stress function Φ such that the membrane forces are given by

$$\begin{aligned}
T_1 &= -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial\Phi}{\partial\alpha_2}\right) - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{1}{A_1}\frac{\partial\Phi}{\partial\alpha_1} \\
T_2 &= -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial\Phi}{\partial\alpha_1}\right) - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{1}{A_2}\frac{\partial\Phi}{\partial\alpha_2} \\
S &= \frac{1}{A_1A_2}\left(\frac{\partial^2\Phi}{\partial\alpha_1\,\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\Phi}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\Phi}{\partial\alpha_2}\right)
\end{aligned} \tag{6}$$

The first two equations of equilibrium (1a) and (1b) can be satisfied approximately when the first-order derivatives of Φ are neglected in comparison with higher order derivatives. Now, the substitution of Eqs. (5) and (6) into Eq. (1c) leads

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to the first Vlasov-type equation:

$$[E\delta^3/12(1-\mu^2)]\Delta\Delta w + (1/1-\mu)\Delta M_T - D\Phi = q \quad (7)$$

where

$$D = \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\left(\frac{1}{R_2}\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{1}{R_1}\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\right)\right]$$

The function Φ is not arbitrary, since it must lead to displacements that are compatible. The condition of compatibility is given by (Ref. 3, p. 28)

$$\frac{k_1}{R_2} + \frac{k_2}{R_1} + \frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left[A_2\frac{\partial\epsilon_2}{\partial\alpha_1} + \frac{\partial A_2}{\partial\alpha_1}(\epsilon_2-\epsilon_1) - \frac{A_1}{2}\frac{\partial w}{\partial\alpha_2} - \frac{\partial A_1}{\partial\alpha_2}\omega\right] + \frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left[A_1\frac{\partial\epsilon_1}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}(\epsilon_1-\epsilon_2) - \frac{A_2}{2}\frac{\partial\omega}{\partial\alpha_1} - \frac{\partial\alpha_1}{\partial A_2}\omega\right]\right\} = 0 \quad (8)$$

Substitution of Eqs. (2, 4, and 6) into Eq. (8) then yields

$$E\delta Dw + \Delta\Delta\Phi + \Delta N_T = 0 \quad (9)$$

where some higher order terms have been neglected. Eqs. (7) and (9), together with appropriate boundary conditions,[4] can be used to obtain approximate solutions for thermoelastic problems of thin shells.

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Total Heating Load on Blunt Axisymmetric Bodies in Low-Density Flow

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Nomenclature

$\dot{q}_0$ = heat transfer rate at stagnation point
$\dot{q}_{0fm}$ = $\rho_\infty U_\infty^3/2$ = approximate stagnation point rate for free molecular flow
$\dot{q}_{avg}$ = average heat transfer rate based on surface area
$\dot{q}_{avg\,fm}$ = average rate for free molecular flow
U_∞ = velocity of freestream
ρ_∞ = density of freestream
R = radius of body ≡ radius of curvature for hemisphere
Re_2 = $\rho_\infty U_\infty R/\mu_2$
μ_2 = viscosity immediately downstream of normal shock
H_0 = total enthalpy of freestream
H_w = enthalpy corresponding to body surface conditions
$(dU/ds)_0$ = "inviscid" velocity gradient at stagnation point

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THIS is a report of total heating rates of blunt, axisymmetric noses in a low-density, hypersonic wind tunnel. Minimum stream density was such that Knudsen number based on nose radius and conditions immediately behind the normal portion of the bow shock exceeded 0.1. Thus, scaling on the basis of Knudsen number, a body of 1-ft nose radius at a maximum altitude above 315,000 ft was simulated.

The LDH wind tunnel,[1] in operation at the von Kármán Gas Dynamics Facility of the Arnold Engineering Development Center, was used to obtain the data presented. Although calorimetry data show thermodynamic equilibrium to exist at the throat of the nozzle, computations[2] show molecular vibration to be essentially frozen downstream of the throat when nitrogen is the medium. A similar calculation indicates that vibration remains frozen throughout the shock layer, but temperatures corresponding to the active and inert degrees of freedom are nearly equal on the stagnation streamline immediately downstream of the bow shock. It is assumed that the portion of the total enthalpy represented by vibration in the present case contributes to heating of the test bodies. Inasmuch as the total temperature was below that at which nitrogen dissociates, no recombination chemistry was involved. When argon was used as the medium, it was assumed that it behaved as a perfect gas, although excitation to a metastable state occurred.

Talbot[3] has shown that the electrical potential of a probe in an ionized stream is an important factor in determining the heat transfer to the probe. As a precaution, all the results presented herein were obtained with the probe grounded with respect to the tunnel wall.

The models consisted of hemisphere-cylinders and flat-faced cylinders. Total heat flux (Btu/sec) to the noses was measured. This was converted to average heat transfer rate per unit area, $\dot{q}_{avg}$ Btu/ft²-sec, by dividing total heat flux by the wetted area of the nose. Descriptions of the models and other details may be found in a test report.[4]

A problem arose in the comparison of the measured average values with theories presented for stagnation point heat transfer because the theoretical distribution appropriate to the flow conditions is not available in all cases. This left no recourse except the assumption that one of the theories for thin boundary layers (high Reynolds numbers) may be used to obtain the relation between average and stagnation point heating rates at very low Reynolds numbers. This was done by assuming that Lees' distribution[5] was valid for the case of the hemispheres. For the case of the flat-nosed models, the distribution computed by Vinokur[6] was used. The relations inferred from these distributions are

Nose shape	$\dot{q}_0$	$\dot{q}_{0fm} = \rho_\infty U_\infty^3/2$
Hemisphere[5]	$2.50\ \dot{q}_{avg}$	$2.00\ \dot{q}_{avg\,fm}$
Flat face[6]	$0.756\ \dot{q}_{avg}$	$\dot{q}_{avg\,fm}$

The experimental results are presented in Fig. 1 and, in the case of the hemisphere, compared with theories for low-density flow.

Behavior of the data appears qualitatively in agreement with results of the most appropriate theories. There is an indication that the data at the lowest values of Re_2 on Fig. 1a depart from the extrapolated, theoretically derived curves. First, it should be noted that the data extend to Reynolds numbers lower than are compatible with the flow models assumed for theoretical analysis. Second, the earlier remarks on the relation of average rates to stagnation point rates may be relevant. The hemisphere tested in argon yielded results in good agreement with theory, as shown in Fig. 1b.

The constant-density, subsonic flow field on which the heating rate distribution of Ref. 6 is based cannot be valid at Reynolds numbers where a fully merged shock layer exists for a highly cooled body. Thus, some of the difference between theory and experiment seen in Fig. 1c would be expected for this reason.